Inside the black hole

Robert W. Brehme
Department of Physics, Wake Forest University, Winston-Salem, North Carolina 27109
(Received 30 June 1976; revised 3 November 1976)

The simplest model of a black hole, the massive point source generating a static spherically symmetric gravitational field, is examined using the Schwarzschild coordinate frame. A brief review is given of this coordinate frame external to the Schwarzschild surface. Greater attention is paid to an interpretation of this frame inside the Schwarzschild surface. Here the roles of space and time are reversed in the sense that the external radial coordinate becomes an internal temporal coordinate, and the external temporal coordinate becomes an internal spatial coordinate. An internal universe is constructed from this frame, and a few simple kinematic phenomena are described in terms of it. The internal and external coordinates are connected graphically by using Kruskal coordinates and physically by considering the world lines of photons and freely moving particles which transit the Schwarzschild surface.

1. INTRODUCTION

If the escape velocity of a particle at the surface of a massive gravitating body is equal to the speed of light, no signal can propagate from the surface to the universe outside the surface in any finite time as measured by clocks fixed in the external universe. Such a body is called a black hole. Particles, including photons, can fall into the body from the outside, but they can never leave.

The possibility of the black hole was first proposed by Laplace[1] in 1795. It is a matter of simple Newtonian mechanics to arrive at a connection between the critical radius and the mass of a body which has become a black hole. Since the escape velocity v from a spherically symmetric gravitating body of radius r and mass M is

$$v^2 = 2GM/r, \tag{1.1}$$

when we set $v = c$, the speed of light, and solve for the corresponding value of r, we obtain

$$R = 2GM/c^2, \tag{1.2}$$

where R is the critical radius of a body which is a black hole.

Again confining the discussion to Newtonian mechanics, we can arrive at an expression for the average density ρ of the body whose radius equals R in terms of its mass:

$$\rho = 3c^6/32\pi G^3 M^2. \tag{1.3}$$

Thus, the larger the mass of the body, the less the average density of the black hole. Setting ρ equal to the density of nuclei, we find that the corresponding critical mass is a little less than 7 solar masses ($7\ M_\odot$). The corresponding radius is about 20 km.

Since stars exist whose masses are greater than $7\ M_\odot$, it seems likely that, if such stars in cooling and contracting shrink to bodies whose radii approach 20 km, black holes will be formed with densities comparable to or less than those of neutron stars. The fact that the radii of such holes are astronomically miniscule probably precludes our detecting them directly, such as by the eclipsing of a normal star. But indirect methods exist which even now indicate the likelihood of the reality of black holes.[2]

The problem of black holes is properly formulated in the language of general relativity and the gravitational theory of Einstein. It is a happy accident that in the simplest models of the black hole general relativity predicts the critical radii and densities to be given by Eqs. (1.1) and (1.2). But the general theory makes the black hole a far more interesting object than does Newtonian theory, for the general theory is concerned with the departure of the character of space–time from the simple Euclidean–Newtonian picture as well as with the character of gravitation.

The purpose of this paper is to examine the general relativistic description of the geometry of space–time within the black hole for the simplest case. While this geometry has been examined many times before, it is the author's hope that this article will present to many readers a new picture of the interior of the black hole which, while very strange indeed, is nevertheless understandable and imaginable.[3]

2. OUTSIDE THE SCHWARZSCHILD SURFACE

In the absence of a gravitational influence, the geometry of space–time is flat in the sense that, if events E_1 and E_2 are infinitesimally close in space and time, an infinite set of coordinate frames exists such that the spatial and temporal coordinate differences, dx, dy, dz, and dt between the events as measured in any one of the coordinate frames, are related by the invariant expression

$$d\sigma^2 = dx^2 + dy^2 + dz^2 - c^2\,dt^2, \tag{2.1}$$

where $d\sigma$ is the *invariant interval* between the events.

The character of space is differentiated from that of time by the presence of the negative sign before the squared temporal interval. If this sign were positive, there would be no physical distinction between space and time. Space would be four-dimensional and matter would be frozen in it.

Gravitational fields warp the fabric of space–time so that Eq. (2.1) is no longer valid. When the gravitating body is a point source of mass M located at the origin of the coordinate frame, Eq. (2.1) can be expressed, under the principles of general relativity, as

$$d\sigma^2 = \frac{dr^2}{1 - R/r} + r^2\,d\Omega^2 - c^2\,dt^2\,(1 - R/r), \tag{2.2}$$

where R is defined by Eq. (1.2). If $M = 0$, Eq. (2.2) reduces to Eq. (2.1) expressed in terms of spherical polar coordinates, in which $d\Omega$ is the element of angle between the spatial locations of events E_1 and E_2, and r is their radial distance from the origin.

Since the source is a point, R does not describe its radius. Nevertheless, R takes on the same kind of meaning here that it had in the simple Newtonian situation. It is a critical length, as measured in the coordinate frame used in Eq. (2.2), which divides the universe into two distinct parts. For values of r greater than R, the universe is very similar to the Newtonian description of space-time at large distances from the source of gravitation. Indeed, as r approaches infinity, Eq. (2.2) reduces to Eq. (2.1) expressed in spherical polar coordinates. However, when r is finite but larger than R, space-time departs from the Euclidean-Newtonian view in the following two important ways.

To illustrate these ways we first pick two nonsimultaneous events which occur at the same *place* in the above coordinated system, so that $dr = 0$ and $d\Omega = 0$. Then

$$d\sigma^2 = -c^2\,d\tau^2 = -c^2\,dt^2\,(1 - R/r). \qquad (2.3)$$

Here $d\tau$ is the *physical* time between the events, as measured say by an atomic clock located at the position of the events. On the other hand, dt is the *coordinate* time between the events. It can be shown that, if the events are observed from a point infinitely remote from the source, the time between them as measured by an atomic clock here is dt. Equation (2.3) gives the well-known gravitational red shift due to the point source. Physical time proceeds more slowly the closer the periodically occurring phenomena are to the source, provided r is greater than R.

Second, we pick two *simultaneous* events which occur at two close but different values of r and along the same radial direction, so that

$$d\sigma^2 = \frac{dr^2}{1 - R/r}. \qquad (2.4)$$

Here $d\sigma$ is the *physical* distance between the events as measured in the spatial coordinate frame, and dr is the difference in the radii of the two concentric circles on which the events lie, which radii are defined according to the Euclidean rule

$$r = \text{circumference}/2\pi. \qquad (2.5)$$

That is, two concentric circles are further apart than the differences in their circumferences would indicate, using the Euclidean relationship in Eq. (2.5).

Hence the general relativistic description of space-time in the neighborhood of a gravitating body differs from that of Newton in that time "slows down" as one approaches the source, and space is "warped" in that the radial distance between circles centered at the source deviates from that specified by Euclidean geometry.

Equation (2.2) describes the geometry of space-time in the presence of a point source. Applying the principle of geodesic motion[4] to this expression yields the equations of motion of objects which move without interactions in this space-time. These equations are interpreted in the general theory as the effect of the gravitating body on the motion of the objects. That is, a geodesic in a curved space-time replaces gravitationally induced accelerated motion in Newtonian space-time. The equations of motion arising from Eq. (2.2) yield the familiar orbital trajectories of Newtonian motion in the limit in which r approaches infinity. For finite r, the Newtonian equations are near approximations of those obtained from general relativity, provided r is still considerably larger than R.

Equation (2.2) is expressed in terms of a particular coordinate system called the *Schwarzschild* coordinates (the S coordinates).[5] This system is useful not only because it reduces easily to the Newtonian approximation for large distances r, but also because the radial coordinate is defined in terms of the Euclidean relationship of Eq. (2.5).

The system is by no means unique. We are free to make transformations to any other set of coordinates we might wish to invent. However, we are cautioned that there is no transformation which removes either the intrinsic curvature of space-time nor the separation of space-time into two distinct parts bounded by the surface $r = R$. The radius of this surface is the radius of the black hole in this simplest model. The surface itself is called the Schwarzschild surface (the S surface).

Equation (2.2) fails when we set $r = R$, for singularities appear which are more a fault of the choice of the coordinate frame than of the geometry itself. Nevertheless, let us examine two events located on the S surface along the same radial direction. It is clear from Eq. (2.2) that these events might yet be separated in space, since $dr^2/(1 - R/r)$ is undetermined. To put it another way, two concentric great circles can exist on the surface, whose circumferences are each $2\pi R$, but which are separated by the finite distance $d\sigma$.

Next, let the two events merge into a single event so that the physical distance $d\sigma$ and the physical time $d\tau$ separating them shrink to zero. However, the S coordinate time between them does not necessarily shrink to zero: since its multiplying factor in Eq. (2.2) is zero, dt can be any finite time. Thus, in S coordinates, the surface itself is frozen in time. This statement does not mean that events cannot be separated in physical time on the S surface. They can be. However, if two events on the surface are separated in a finite physical time, only one of them can occur in the present of the S observer outside the surface. The other occurs either in the infinite past or infinite future.

3. INSIDE THE SCHWARZSCHILD SURFACE

When r is less than R, a remarkable change occurs in the nature of space-time when viewed in the S framework. The signs before the squared temporal and radial intervals in Eq. (2.2) are reversed. Therefore, what to the outside S observer is a radial coordinate becomes to the inside S observer a temporal coordinate. Similarly, the temporal coordinate for the outside S observer becomes a spatial coordinate for the inside S observer.

The character of the universe within the S surface is better understood if we adopt a nomenclature and symbolism which reflect the physical properties of the coordinates. Thus, in place of r in Eq. (2.2) we write ct, and in place of ct we write z. Equation (2.2) becomes under this "transformation"

$$d\sigma^2 = dz^2\,(T/t - 1) + c^2t^2\,d\Omega^2 - \frac{c^2\,dt^2}{T/t - 1}, \qquad (3.1)$$

where

$$T = R/c. \qquad (3.2)$$

It was pointed out above that there is no one given coordinate frame which must be used to describe space-time.

We shall have occasion later, for example, to introduce the Kruskal coordinates in order to connect the geometries within and without the S surface. But the slightly transformed system given above produces a physically meaningful, though strange, picture of the universe inside the surface, and we shall consider its implications in some detail.

One property of this frame, which derives from the principle of geodetic motion, is that the equations of motion of noninteracting objects yield solutions in which objects at rest remain at rest. This situation is unlike that which holds outside the surface, where objects are accelerated (relative to the S frame of reference) toward the origin. However, this situation does not mean that objects behave as though they were in an inertial frame of reference. For if Eq. (3.1) is applied to two objects at rest and separated at the same coordinate instant by dz, the coordinate interval dz remains constant in coordinate time, but the *physical distance* $d\sigma$ between the objects changes in coordinate time according to the equation

$$d\sigma = dz(T/t - 1)^{1/2}. \tag{3.3}$$

That is, as the coordinate time advances, the physical distance separating the noninteracting objects in the z direction grows smaller.

Next let us consider Eq. (3.1) as it applies to two objects at rest and separated at the same coordinate instant by $d\Omega$. The coordinate interval $d\Omega$ remains constant, but the physical distance between the objects changes in coordinate time according to the equation

$$d\sigma = ct\, d\Omega. \tag{3.4}$$

That is, as the coordinate time advances, the physical distance separating the noninteracting objects in the Ω surface grows larger in direct proportion to the coordinate time (a fact which can be used to measure coordinate time).

The relationship between the z axis and the Ω surface is evident from Eq. (3.1). The fact that the expression for $d\sigma^2$ involves only the sums of the squares of dz and $d\Omega$ implies that all displacements given by Eq. (3.3) are physically perpendicular to all displacements given by Eq. (3.4). The fact that Eq. (3.4) is independent of z means that all displacements given by Eq. (3.3) and originating at different points in the Ω surface are parallel to one another. In this sense then, the Ω surface is a plane. However, the spatial geometry of this surface remains that of the surface of a Euclidean sphere of radius ct.

Thus, the universe within the S surface is closed but unbounded in any direction on the Ω surface. The "radius" of this surface grows in coordinate time at the rate c, and its physical area increases at the rate $8\pi c^2 t$. On the other hand, the interior universe in the z direction is infinite.

We next investigate the behavior of clocks which remain at rest in this coordinate frame. Equation (3.1) shows that the physical time, $d\tau$, kept by such clocks is related to the coordinate time, dt, according to

$$c^2\, d\tau^2 = \frac{c^2\, dt^2}{T/t - 1}. \tag{3.5}$$

The positive root of this expression can be integrated between the limits $t = 0$ and $t = T$ to give

$$\tau = T\{\pi/2 - \arccos(t/T)^{1/2} - [(t/T)(1 - t/T)]^{1/2}\}. \tag{3.6a}$$

The constant of integration has been chosen so that $\tau = 0$ when $t = 0$. When $t = T$, $\tau = \pi T/2$. Equation (3.6a) shows that, while the relation between coordinate time and the physical time kept by clocks at rest in this frame is not simple, neither is it pathological nor grotesque. Coordinate time advances in one-to-one correspondence with the physical time kept by rest clocks.

The negative root of Eq. (3.5) can also be integrated to yield

$$\tau = T\{\pi/2 + \arccos(t/T)^{1/2} + [(t/T)(1 - t/T)]^{1/2}\}, \tag{3.6b}$$

so that when $t = T$, $\tau = \pi T/2$, and when $t = 0$, $\tau = \pi T$. Thus, as coordinate time decreases, the physical time kept by rest clocks increases.

It is clear that Eq. (3.1) allows two kinds of interior worlds. In one, objects at rest in the frame move toward one another in the z direction and fly apart on the Ω surface. In the other, objects at rest fly apart along the z direction and move toward one another on the Ω surface.

In both interior worlds the universe has a finite lifetime. Measured by coordinate clocks it is T, and by rest clocks it is $\pi T/2$. The evolution of these universes is accompanied by the following development of its geometry.

Starting with Eq. (3.6a) and $t = 0$, we have an interior universe collapsed upon the z axis, with no extension into the other two spatial dimensions. A singularity occurs in the z coordinate in that, if any two events are separated by a finite distance $\Delta\sigma$, their z coordinates are the same. Objects which at the beginning of this universe are a finite distance apart immediately collapse together as the universe ages. In order that such a collapse not occur, it is necessary that, initially, the objects be infinitely remote from one another. Unlike the situation which occurs at the S surface, this pathological behavior is inherent in the structure of space–time. It cannot be removed by a change in coordinate frame.

The "point" source of Eq. (2.1) is not a point source in the framework used here. Rather it is an "instant" source. The gravitational mass M appears only at the moment $t = 0$, distributed uniformly along the z axis. For all later moments it has vanished. That is, the universe described by Eq. (3.1) is established by an initial condition rather than by a boundary condition in space.

As coordinate time advances, matched by a corresponding advance in the physical time of rest clocks, "rest" objects move together along the z axis and spread out across the Ω surface. The process continues until $t = T$, at which moment the spatial dimension along the z direction has vanished. All objects now lie only in the two-dimensional Ω surface, which has reached its maximum area.

We may further the evolution of the interior universe by picking up Eq. (3.6b) and allowing coordinate time to run backwards while the physical time of the rest clocks continues its inexorable march forward. The spatial z dimension reappears perpendicular to the Ω surface, and objects move away from one another in this direction while collapsing together along the Ω surface. Finally, when $t = 0$ and $\tau = \pi T$, the interior universe comes to a halt. Objects once again are stretched along the z axis, infinitely remote from one another, and the gravitational mass which disappeared after the instant of beginning reappears at the instant of ending.

Up to now we have considered only objects at rest in the interior world. The principle of geodesic motion applied to Eq. (3.1) also yields solutions which describe objects moving freely relative to the coordinate frame. For a particle restricted to the z axis but otherwise moving freely, the equation for the *local proper speed* of the particle is

$$\frac{d\sigma}{d\tau}=\frac{U}{(T/t-1)^{1/2}}. \quad (3.7a)$$

Here U is a constant of integration, $d\sigma$ is the physical distance traveled along the z axis, and $d\tau$ is the lapse of physical time on the particle. At $t = 0$, the speed of the particle is zero and at $t = T$, the speed is infinite. (The local proper speed of light is always infinite, but it is not correct to say, as we shall see in the next section, that the particle moves at the speed of light.)

For a particle restricted to the Ω surface but otherwise moving freely, the equation for the local proper speed of the particle is

$$\frac{d\sigma}{d\tau}=\frac{cT_0}{t}, \quad (3.7b)$$

where T_0 is a constant of integration. At the moment of creation the speed of the particle is infinite. But as time advances, its speed drops until at the moment of greatest expansion in the Ω surface, its speed is cT_0/T.

Since the interior universe along the Ω surface is finite in extent, we wonder if it is possible to circumnavigate the interior universe in its lifetime. Since the speed of light cannot be exceeded by any physical particle, we shall examine the time taken for light to make one circuit of the interior universe and draw our conclusions from the result.

For light signals, $d\sigma = 0$ in Eq. (3.1). Letting $dz = 0$ in Eq. (3.1), we have for the light signal confined to the Ω surface

$$\frac{d\Omega}{d\tau}=\frac{1}{[t(T-t)]^{1/2}}, \quad (3.8)$$

in which we have chosen the positive root. The integral of this equation is

$$t/T=\sin^2(\Omega/2), \quad (3.9)$$

where the constant of integration has been chosen so that, when $\Omega = 0$, $t = 0$. Equation (3.9) shows that a light signal which starts moving in the Ω surface at $t = 0$ arrives at $\Omega = \pi$ rad at $t = T$. Thus the light signal moves only halfway around the universe during its period of Ω expansion. However, it may move the other half of the distance during the period of contraction, thereby exactly circumnavigating the interior universe during its entire lifetime. A material particle would not be able to complete the journey in the time allowed.

4. KRUSKAL COORDINATES: A GOD'S-EYE VIEW

The S coordinate system, while suitable for giving limited physical views of space–time inside and outside the S surface, fails us if we attempt to understand what connection, if any, can exist between the interior and exterior universes. What happens, for example, to an object which falls or is fired into the surface from the outside? Events on the surface are frozen in time for all outside S observers, and so clearly no object can ever penetrate the surface from the point of view of the outside S observer. But is this limitation a truly physical one, or does it arise because of an unsuitable choice of coordinate frame?

It develops that the two universes can be connected by objects passing back and forth through the S surface and that coordinate frames can be used in which the singularities found in the S coordinate frame vanish at the surface. One such frame is that using *Kruskal* (K) coordinates.[6]

The K transformation maps the S coordinates r and t (outside) and z and t (inside) onto the unitless coordinates p and q as follows:

$$q^2=(r/R-1)\exp(r/R)\cosh^2(ct/2R), \quad (4.1a)$$

$$p^2=(r/R-1)\exp(r/R)\sinh^2(ct/2R), \quad (4.1b)$$

which apply to the exterior universe;

$$q^2=(1-t/T)\exp(t/T)\sinh^2(z/2cT), \quad (4.1c)$$

$$p^2=(1-t/T)\exp(t/T)\cosh^2(z/2cT), \quad (4.1d)$$

which apply to the interior universe. For the exterior universe the inverse transformations are

$$q^2-p^2=(r/R-1)\exp(r/R), \quad (4.2a)$$

$$p/q=\tanh(ct/2R), \quad (4.2b)$$

and for the interior universe

$$p^2-q^2=(1-t/T)\exp(t/T), \quad (4.2c)$$

$$q/p=\tanh(z/2cT). \quad (4.2d)$$

Under these transformations, the expression for the invariant interval of Eqs. (2.2) and (3.1) for the exterior universe becomes

$$d\sigma^2=(4R^3/r)\exp(-r/R)(dq^2-dp^2)+r^2\,d\Omega^2, \quad (4.3a)$$

and for the interior universe becomes

$$d\sigma^2=c^2(4T^3/t)\exp(-t/T)(dq^2-dp^2)+c^2t^2\,d\Omega^2. \quad (4.3b)$$

Suppressing the spatial dimensions lying in the Ω surface, we can plot curves of constant r and t (outside) and z and t (inside) in a rectangular p,q plane, as shown in Fig. 1. This is the familiar diagram displaying the relationship between K and S coordinates.

For the exterior universe, events of constant r lie on rectangular hyperbolas and events of constant t lie on straight lines through the origin. For the interior universe, the roles of space and time are interchanged. Events of constant z lie on straight lines through the origin and events of constant t lie on rectangular hyperbolas.

In terms of the exterior S frame, the origin of Fig. 1 is the event on the S surface existing at all finite times. The asymptotes of the hyperbolas are the locus of events on the S surface whch occur either in the infinite past or infinite future. But in terms of the interior S frame, the origin is the event occurring at the moment of complete collapse of space in the z direction, at $t = T$. And the asymptotes are the locus of events lying at the infinite extremes of the z axis and occurring at $t = T$.

The equation of motion of a light signal in K coordinates is found by setting either of Eqs. (4.3) equal to zero. Again suppressing the spatial dimensions lying in the Ω surface, we obtain

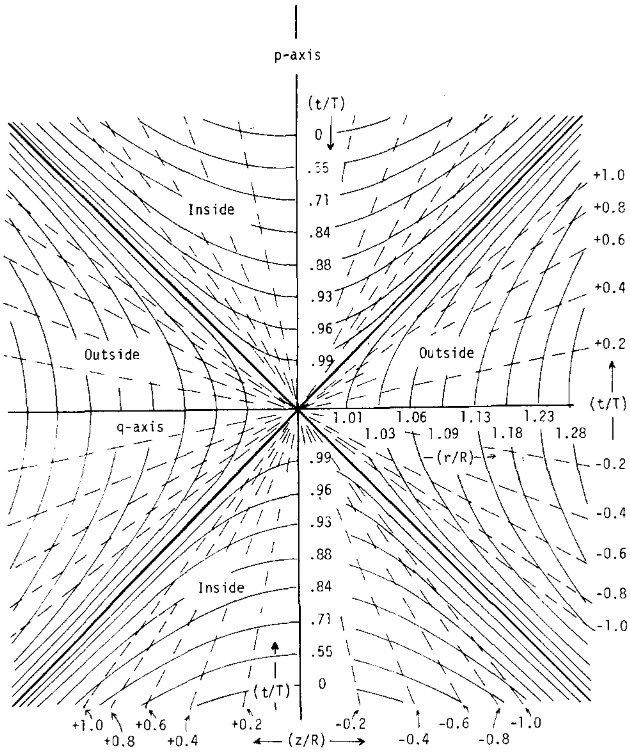


Fig. 1. In the rectangular frame of Kruskal coordinates, the curves of constant r and t (exterior) are respectively rectangular hyperbolas and straight lines radiating from the origin. Similarly, the curves of constant t and z (interior) are hyperbolas and straight lines. These curves are drawn to scale in the unitless quantities of r/R, t/T (exterior), and z/R, t/T(interior).

$$dp^2 = dq^2. \tag{4.4}$$

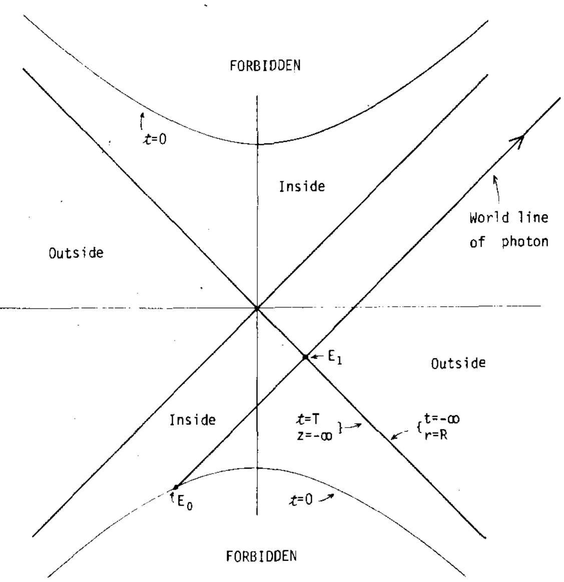
That is, the locus of events describing any photon is a straight line in the p,q plane parallel to the asymptotes. An example of such a locus, or world line, is shown in Fig. 2(a). A photon is created in the interior world at event E_0 at $t = 0$ at a point on the z axis, say, equal to $+2R$. The light propagates along the z axis in the negative direction at the local speed of light, arriving at $z = -\infty$ at $t = T$. This event, E_1, appears in the external S frame as occurring on the surface in the infinite past. The light climbs out from the surface and forever moves away from the gravitational source.

It is clear that light originating in the interior universe can enter the exterior universe. This light always leaves the interior universe at one of two "places" ($z = \pm\infty$) at the same time ($t = T$) in the view of the interior S observer. It always appears in the exterior universe at the same place ($r = R$) in the infinite past of the exterior S observer.

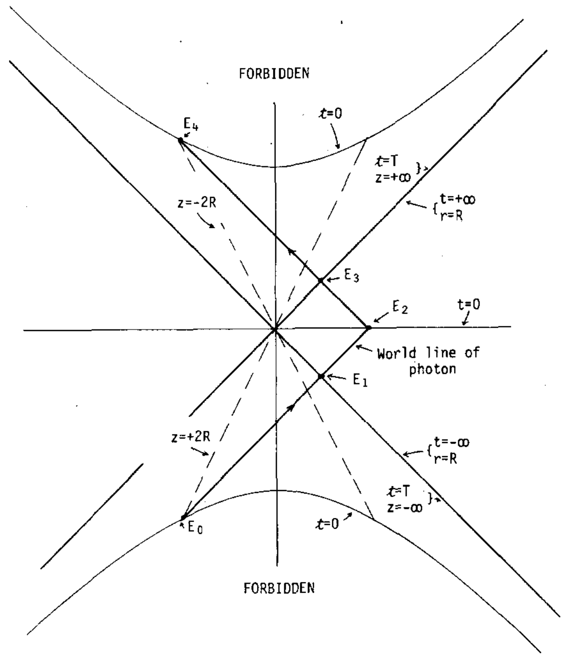
A second example of the world line of a photon is shown in Fig. 2(b). Here the photon is reflected back toward the gravitational source at event E_2, falling into the S surface in the infinite future of the exterior frame at event E_3. This event is marked in the interior S frame as the reappearance of the photon on the z axis at positive infinity at the moment $t = T$. Thus, from the point of view of the interior S observer, the light disappears from the negative side of the collapsed z axis at the moment of complete contraction only instantly to reappear on the positive side of the z axis, moving in the same negative direction. No time is allowed in the interior S frame for the journey of the photon in the exterior universe.

The photon continues propagating along the z axis in the negative direction, arriving at the end of the interior universe at E_4, when $t = 0$ and $z = -2R$.

Photons are not the only particles which may pass from one side of the S surface to the other. An example is shown in Fig. 3. Here a material particle is created at event E_0 at the beginning of the interior universe. Although its initial local proper speed is zero, its world line is governed by Eq.



(a)



(b)

Fig. 2. (a) A photon is created at E_0 ($t = 0$, $z = 2R$) in the interior world and moves along the negative z axis. It vanishes from the interior world at E_1 ($t = T$, $z = -\infty$) and enters the exterior world at E_1 ($t = -\infty$, $r = R$) and remains in the exterior world. (The diagram is not drawn to scale.) (b) A photon is created at E_0 ($t = 0$, $z = 2R$) in the interior world and moves along the negative z axis. It vanishes from the interior world at E_1 ($t = T$, $z = -\infty$) and appears in the exterior world at E_1 ($t = -\infty$, $r = R$). It is reflected at E_2 ($t = 0$, $r = r_1$) and arrives at the Schwarzschild surface at E_3 ($t = +\infty$, $r = R$). It enters the interior world at E_3 ($t = T$, $z = +\infty$) and moves in the negative direction along the positive z axis, arriving at event E_4 ($t = 0$, $z = -2R$), where it is destroyed. (The diagram is not drawn to scale.)

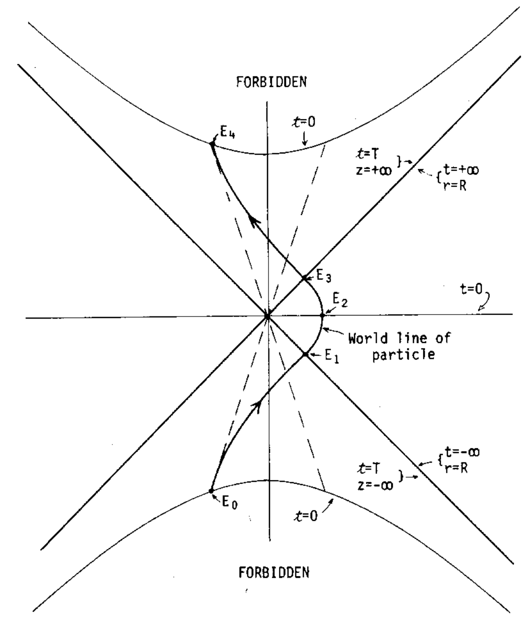


Fig. 3. A particle is created at E_0 in the interior world and moves along the negative z axis at an ever increasing local speed. It vanishes from the interior world at E_1, moving at something less than the speed of light, and appears in the exterior world at the same event. At event E_2 it is at its maximal displacement from the black hole. At event E_3 it has returned to the Schwarzschild surface and enters the interior world. It moves in the negative z direction and is destroyed at event E_4. (The diagram is not drawn to scale.)

(3.7a), the equation of motion of a body freely falling along the z axis. It arrives at the S surface at event E_1, moving at an infinite speed in the interior frame. However, an analysis of the motion relative to the K coordinates shows that the slope of its world line at the S surface is greater than unity. That is, it transits the surface at less than the speed of light. Thus it enters the exterior world in the infinitely remote past in the S frame with an escape velocity which determines its maximal radial displacement from the gravitational source. In Fig. 3, this extreme displacement is finite and occurs at event E_2. The particle then falls back toward the S surface, arriving in the infinite future in the S frame at event E_3.

Like the photon above, it reappears in the interior world at the positive end of the z axis at the moment of total collapse of the z axis. It continues moving along the z axis until it is destroyed at event E_4.

Unlike the photon, the material particle may register the lapse of time relative to itself, its so-called proper time. It can be shown that the lapse of proper time for the journey from E_0 to E_1 is given by

$$\tau/T = k^{-3/2}\{\arccos(kT/T)^{1/2} - [(kt/T)(1 - kt/T)]^{1/2}\}, \qquad (4.5a)$$

where $k = 1 - (U/c)^2$, and $U < c$. The proper time for the journey from E_1 to E_2 is given by

$$\tau/T = (R_0/R)^{3/2}\{\arccos(r/R_0)^{1/2} + [(r/R_0)(1 - r/R_0)]^{1/2}\}, \qquad (4.5b)$$

where $R_0 = R/[1 - (U/c)^2]$, and $U < c$. Since the motion is symmetric about event E_2, doubling the times given in Eqs. (4.5) yields the total time.

Although the exterior S observer judges the lifetime of the particle in the exterior world to be infinite, the lapse of time on the particle is finite. And although the interior S observer judges the particle instantly to disappear and reappear at the extremes of the z axis, a clock carried by the particle indicates a lapse of time which the interior observer cannot account for in terms of his frame of reference.

It is emphasized that the foregoing is a model of a *static* black hole due to a point source which has existed for all external S time in an otherwise empty universe. Real black holes may have been present at the moment of creation of the real universe, if such ever occurred, and may now be still with us. Or they may evolve from the gravitational collapse of massive stars, in which case their properties are very different from those of the simple example examined here. Nevertheless, this model presents a fascinating picture of the way in which the general theory of relativity alters the Euclidean–Newtonian fabric of space-time in an extraordinary but comprehensible way.

[1]P. S. Laplace, *Le Système du monde* (Paris, 1795), Vol. 2, p. 305; C. Misner, K. Thorne, and J. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), pp. 623 and 872.

[2]The best known candidate for a black hole is the x-ray source, Cygnus X-1, a double star whose luminous member is a blue giant (spectral classification B, on the main sequence). The invisible member has a mass calculated to be at least 8 $M_{\odot}$ and is the source of x rays. These are assumed to be generated as gasses surrounding the source are drawn into the source and compressed. See Ref. 1, Misner *et al.*, p. 885.

[3]The black hole is amply treated in the literature, and no attempt is made here to refer to all articles on the subject. The few citations here contain thorough bibliographies: R. Ruffini and J. Wheeler, Phys. Today **24** (12), 30 (1971); P. C. Peters, Am. Sci. **62**, 575 (1974); Ref. 1, Misner *et al.*, Chap. 33.

[4]For an excellent discussion of the principle of geodesic motion at the introductory level, see K. Ford, *Classical and Modern Physics* (Xerox College Publishing, Lexington, MA, 1974), Vol. 3, p. 1103 ff.

[5]K. Schwarzschild, Dtsch. Akad. Wiss. Berlin, 189–196 (1916); C. Møller, *The Theory of Relativity* (Oxford U. P., London, 1952), p. 324 ff; J. Anderson, *Principles of Relativity Physics* (Academic, New York, 1967), p. 381 ff.

[6]M. D. Kruskal, Phys. Rev. **119**, 1743 (1960); G. Szekeres, Publ. Mat. Debrecen **7**, 285 (1960); Ref. 1, Misner *et al.*, p. 827 ff.